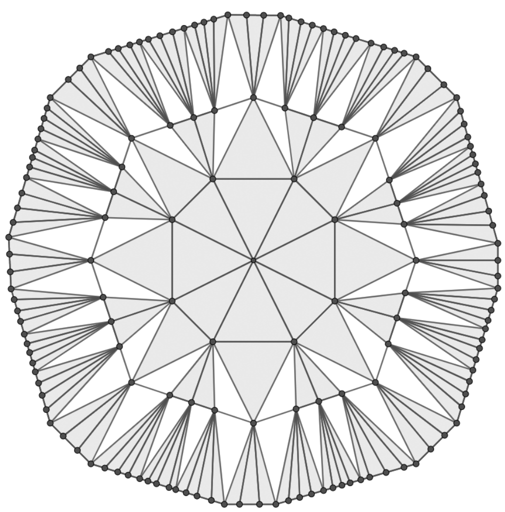
University of North Carolina at Asheville

Geometry Portfolio

Selected Proofs and Reflections on Geometry



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1 Graphs give rise to metric spaces

A metric space is a set with a function d that assigns a metric (or "distance") between any two elements of that set. A metric space must satisfy these properties:

1. $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) + d(y, z) \geq d(x, z)$

A graph structure is defined as $G = (V, E)$, where a collection of vertices V are connected by edges E. We can reach adjacent vertices from any vertex by following the edges that connect them to other vertices. We will only consider connected graphs, where it is possible to reach all vertices starting from any vertex. The distance function d, for any two vertices, is defined by the least number of edges required to travel from one vertex to the other.

Proof. In order to prove that a connected graph is a metric space, we need to prove that the above 3 properties hold.

1. In order to calculate d, we start at count 0, and only increase that count for each edge traveled while moving to the destination. Therefore, d must be greater than or equal to 0. Furthermore, the only way for the count to be 0 would be if we didn't travel along any edges, which would increase the count. If we don't travel along any edges, then we must have stayed on the starting vertex. And, if we stay on the same vertex, then we haven't increased the count by moving along any edges. Therefore $d(x, y) = 0 \iff x = y$.

2. The same edges between x and y are between y and x, and the edges forming the shortest path are also the same. Say that the collection of edges $\{e_1, e_2, ..., e_n\}$ denotes a shortest path from x to y. Then the same set of edges reversed, $\{e_n, e_{n-1}, ..., e_1\}$, will be a shortest path from y to x. There may be multiple such paths, but their count will be the same. Thus, $d(x, y) = d(y, x)$.

3. Consider that

   a. If $d(x, y) + d(y, z) < d(x, z)$, then $d(x, z)$ is not the shortest path.

   The path from x to y and then from y to z would be shorter. We can already stop here and assert that it's impossible for $d(x, y) + d(y, z) < d(x, z)$. In order to prove that the inverse of the equality is not simply vacuously true, i.e. that $d(x, y) + d(y, z) \geq d(x, z)$, consider that

   b. If $d(x, y) + d(y, z) > d(x, z)$, then that means that y is simply off of the shortest path between (x, z). This can only add distance to the left side of the inequality, as it's impossible for a distance to be negative, as shown in (1).

   Therefore $d(x, y) + d(y, z)$ must be greater than or equal to $d(x, z)$.

Since these three properties have been satisfied, we can conclude that a graph gives rise to a metric space. □

2 Exploring the combinatorics of a triangle tiling

See Figure 1 for a tiling of triangles with 8 triangles met at each vertex, extended for 3 spheres. Following from the previous section, this structure can be modeled as a graph, but we won't be looking at its metric properties.

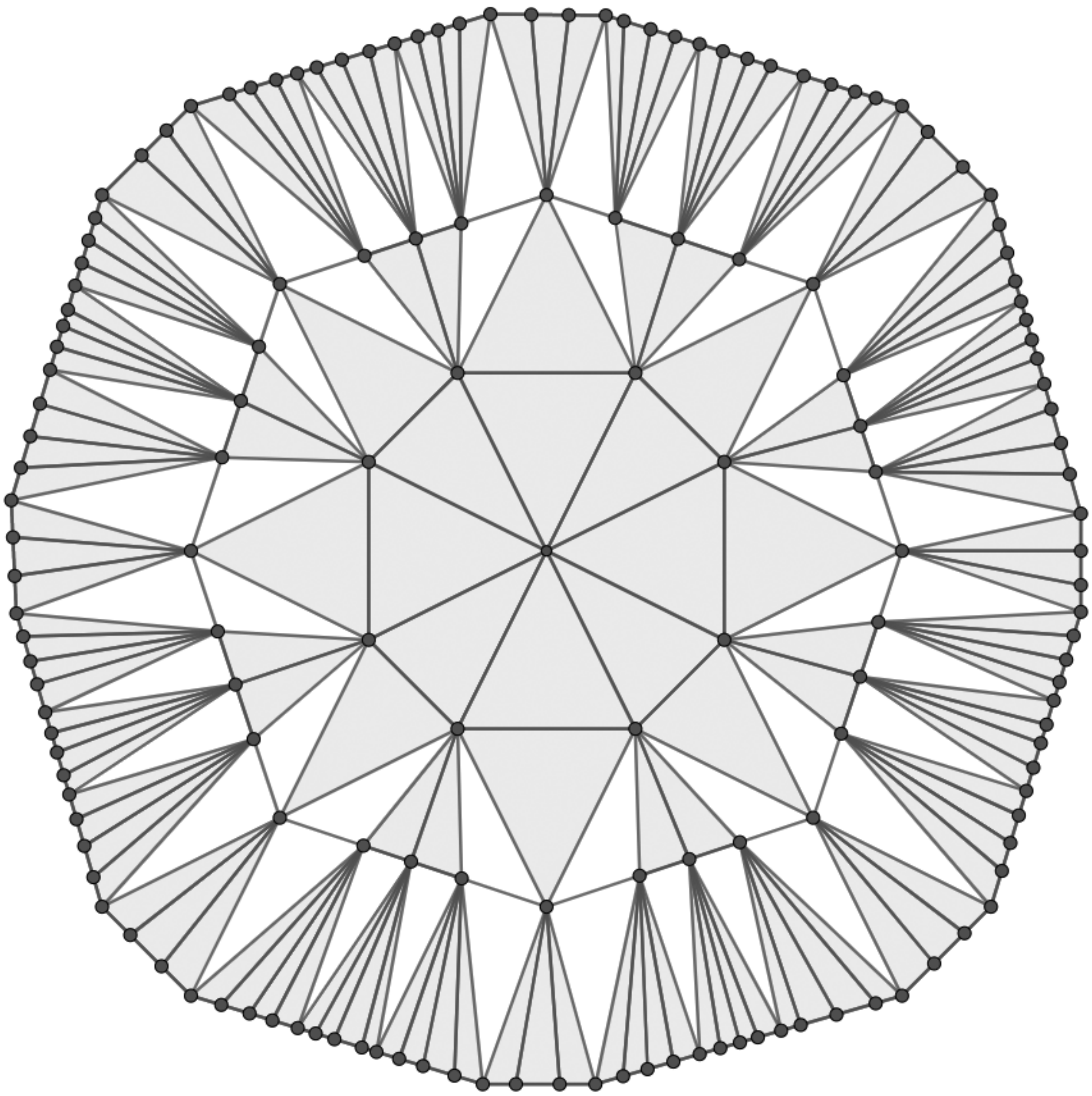

Figure 1: 8 triangle tiling to S_3.

Instead, we are looking at two types of vertices: vertices with just one edge pointing back towards center vertex v_0 (type 1), and vertices with two edges pointing back toward v_0 (type 2). From Figure 1, we count the amount of each type of vertex in the first 3 layers:

n	s_n (type 1)	t_n (type 2)
0	n/a	n/a
1	8	0
2	24	8
3	88	33

From this, we can generalize recursive formulas for the amounts of each type of edge in succeeding layers:

$$s_n = 3s_{n-1} + 2t_{n-1}$$

$$t_n = s_{n-1} + t_{n-1}$$

We use these formulas to calculate s_4, s_5, t_4, and t_5 below.

We can then use these formulas to calculate the number of edges contained by a sphere/spherical shell, denoted as $S_n(v_0)$, as well as a solid cloud/ball, denoted as $B_n(v_0)$. We can get $|S_n(v_0)|$ by combining both types of edges, $s_n + t_n$, for each n. To get $|B_n(v_0)|$, we must add in all prior layers, which are included by the solid sphere.

n	s_n (type 1)	t_n (type 2)	$\lvert S_n(v_0)\rvert = s_n + t_n$	$\lvert B_n(v_0)\rvert = \sum_{k=0}^{n} \lvert S_k(v_0)\rvert$
0	n/a	n/a	1	1
1	8	0	8	9
2	24	8	32	41
3	88	33	120	161
4	328	120	448	609
5	1224	448	1672	2281

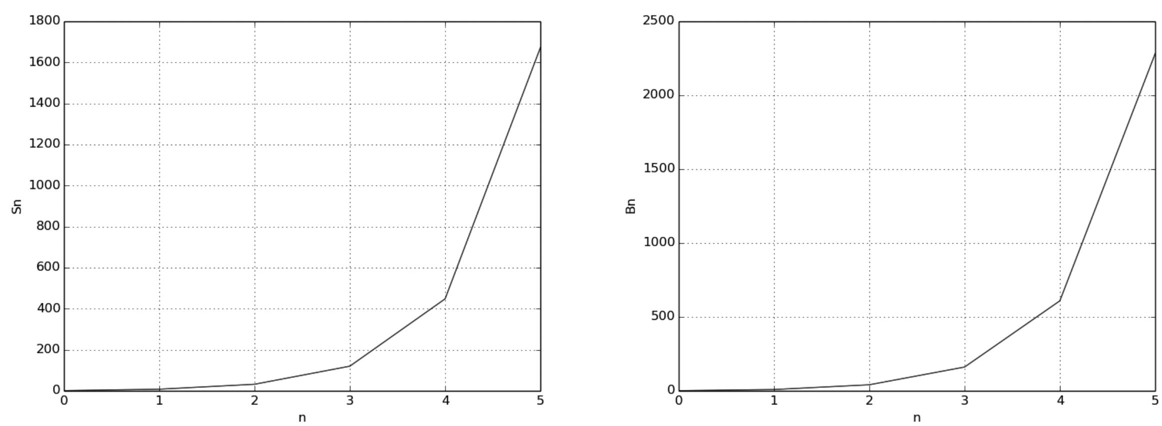

Figure 2: S_n and B_n as n increases look exponential.

Figure 2 shows the last two columns plotted out. From this, they appear to be exponential, which would make them $\mathcal{O}(n^2)$.

Now we wish to find a general formula for tiling with k number of triangles. The formulas for the amounts of each type of vertices in a tiling of 7 triangles were

$$s_{n,7} = 2s_{n-1} + t_{n-1}$$
$$t_{n,7} = s_{n-1} + t_{n-1}$$

Comparing these with the formulas for a tiling of 8 triangles, it's apparent that t_n remains the same for both values of k. And the difference for s_n appears to be varying the coefficients by a factor of 1. We can hypothesize a general formula for when $k \geq 7$:

$$s_{n,k} = (k-5)s_{n-1} + (k-6)t_{n-1}$$
$$t_{n,k} = s_{n-1} + t_{n-1}$$

For very large values of k, $s_{n,k}$ becomes much larger than $t_{n,k}$, k times larger. In other words, $\lim_{k\to\infty} s_{n,k} = k * t_{n,k}$. This is supported by Figure 2, as the shape of the graph looks the same; only the scale is changed.

This structure can be seen as a model of hyperbolic space. In this space, if we define a line as a path on the graph that doesn't ever cross itself, there are infinitely many parallel lines passing between any two points, a sort of reversal of the Euclidean parallel property (this should be easily envisioned, looking at Figure 1). Below, we will prove a simple result in hyperbolic geometry, one in a series showing that the axioms other than the parallel postulate do transfer between the geometries.

3 Verifying Hilbert's betweenness axiom 3 for Klein's model

We wish to show that this axiom follows from Hilbert's Euclidean geometry into Klein's hyperbolic geometry:

Betweenness Axiom 3: If A, B, and C are three distinct points lying on the same line, then one and only one of the points is between the other two. [Greenberg]

Proof. Suppose we have points P, Q, and S that are all distinct and lying on the same line ℓ in the hyperbolic plane. This can be seen in Figure 3 below.

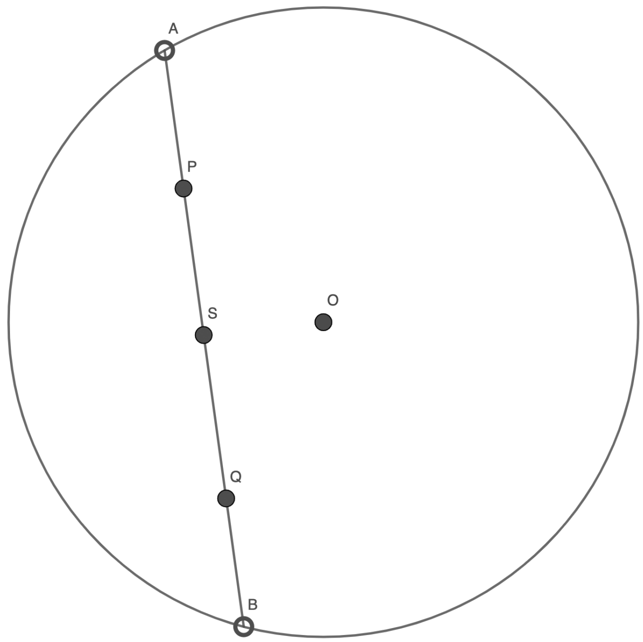

Figure 3: Line in hyperbolic

From our hypothesis, we know $P * S * Q$. By way of contradiction, suppose $P * Q * S$. Thus, we have $P * S * Q$ and $P * Q * S$. However, if we extend the line into the Euclidean plane (Figure 4), this violates the definition of betweenness in the Euclidean plane.

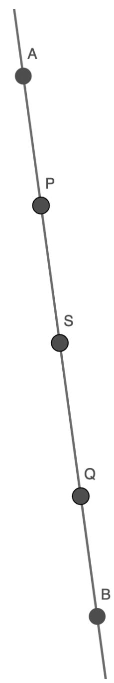

Figure 4: Line in Euclidean

Since the hyperbolic plane inherits the betweenness definition used in the Euclidean plane, $P * S * Q$ and $P * Q * S$ is a contradiction. Therefore, betweenness axiom 3 holds in the hyperbolic plane. □

4 Incidence axioms are independent

Now we will show that the three axioms of the very simple incidence geometry (which can be defined in set theory as sets of points, lines, and 'incidences') are *independent* - that is, any one of them cannot be proven from the others.

Incidence Axiom 1: For every point P and for every point Q not equal to P, there exists a unique line l incident with P and Q.
Incidence Axiom 2: For every line l, there exist at least two distinct points incident with l.
Incidence Axiom 3: There exist three distinct points with the property that no line is incident with all three of them. [Greenberg]

Proof. To prove the axioms are independent, we will create incidence geometries in which two axioms hold but one does not.

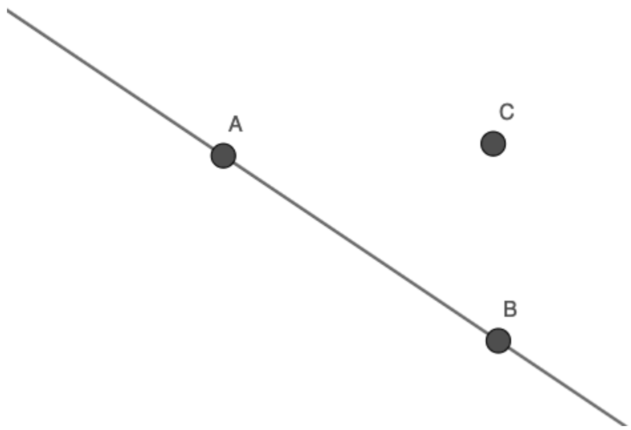

Figure 5: 2&3 but not 1.

This geometry satisfies axiom 2 & 3 but not 1: There are three points, A, B, and C, and one line $\{A, B\}$. This axiom doesn't satisfy axiom 1 because there are no lines between A and C or B and C. Axiom 2 is satisfied since the one line $\{A, B\}$ is incident with the two points A and B. Axiom 3 is satisfied because of the three points, A and B are on the same line, but C is not.

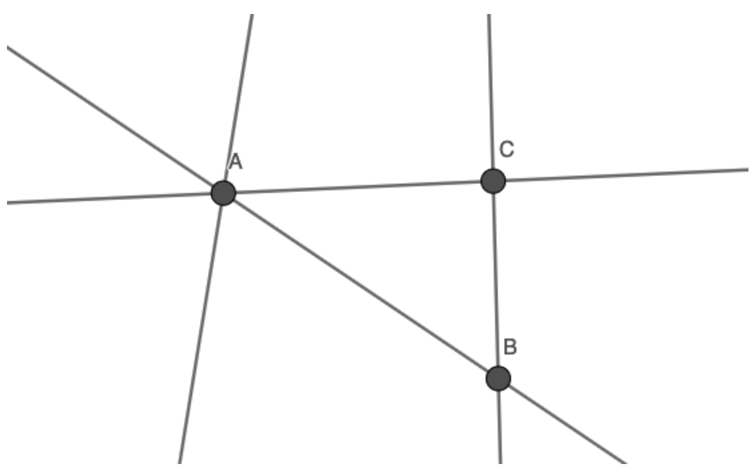

Figure 6: 1&3 but not 2.

This geometry satisfies axiom 1 & 3 but not 2: There are three points, A, B, and C, and four lines $\{A, B\}$, $\{A, C\}$, $\{B, C\}$, and $\{A\}$. Axiom 1 is satisfied because there is a line (all lines except the last one listed) going through every pair of points. Axiom 2 is not satisfied because line $\{A\}$ only goes through one distinct point. Axiom 3 is satisfied because no one line goes through A, B, and C.

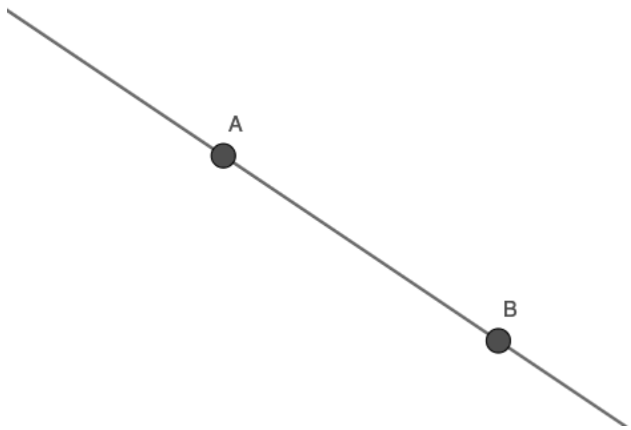

Figure 7: 1&2 but not 3.

This geometry satisfies axiom 1 & 2 but not 3: There are two points, A and B, and one line, $\{A, B\}$. This satisfies axiom 1 because there is a line between the only two points. This satisfies axiom 2 because the one line has two points incident with it. There are only two points, and axiom 3 requires there to exist at least 3 points, therefore axiom 3 is not met.

Therefore, since each axiom can be made false while the others remain true, the three incidence axioms are independent. □

5 Reflection on non-traditional mathematics

I once thought that traditional, Western mathematics was a universal bastion of truth, that these mathematical ideas were so pure that they were discovered facets of the universe itself, called a "transcendent Platonic mathematics" by Lakoff in *Where Mathematics Comes From.* I now realize that math is completely a human invention.

The shaky ground is apparent when learning about geometry. There is a necessity of undefined terms, and the starting axioms have to be taken for granted, and, worse, completely contradictory axioms can make just as much "sense."

The fifth postulate, for example, is not obvious. Non-Euclidean geometry isn't unimaginably alien. The curvature of space could make parallel lines intersect at infinity, as it does in some geometries. Space could loop back on itself at the edge of the universe, making elliptical geometry the actually correct type of geometry to apply to our universe, at least at that scale. This is similar to the difference between Newtonian and Einsteinian physics; Newtonian works well for objects on our scale, but when you look at the big picture, it falls apart. These all involve looking from the human perspective in the universe, which we are fundamentally limited by.

Lakoff argues that the only math we can know is "*mind based mathematics*, limited and structured by human brains and minds." Math is all metaphor, which is only done by brains.

Math produces interesting results, that is all. So, why shouldn't other forms of knowledge be treated equally? As described in Betasamosake Simpson's *Land as pedagogy*, the Native American Nishnaabeg learned how to produce maple syrup by seeing squirrels in the trees. Shouldn't learning by observing nature, which has existed and thrived for billions of years before humans invented math, also produce interesting results?

6 Portfolio reflection

I start out with **Graphs give rise to metric spaces** because I think the two topics, graphs and metric spaces, are very interesting. Graphs are familiar from the computer science discipline, and metric spaces seem like they enable for a lot of other parts of geometry in the familiar plane, by defining distance. The idea of a metric is simple and abstract, and I thought it was interesting how it tied distances on a graph to distances on a plane, two seemingly unrelated but important concepts.

Using the concept of graphs, this leads naturally into **Exploring the combinatorics of a triangle tiling**. This piece is probably my favorite in the portfolio and the class. It involves finding a bunch of fun results about an interesting piece of geometry, as well as seeing how fast the numbers grow in hyperbolic geometry. This leads to a discussion of the parallel postulate in hyperbolic geometry.

All of the axioms but the parallel postulate are the same in Euclidean and hyperbolic geometry, under Hilbert's and Klein's models, respectively. The next proof, **Verifying Hilbert's betweenness axiom 3 for Klein's model**, shows how one of the axioms transfers between the geometries.

I debated putting **Incidence axioms are independent** at the start, since it is naturally the simplest and the starting point of the course, but I wanted to start with something more interesting. Also, this simple discussion of these somewhat rickety-feeling foundations leads into the next section.

In **Reflection on non-traditional mathematics**, I discussed how, without the existence of one right truth, all forms of human knowledge are equal, so we should learn more from each other and from other cultures.

For my oral presentation portion, I have attached **Modern advancements in navigation**, a discussion of technologies for navigation. From this, I determined a common thread of time, a very human concept, as well as different geometric methods, being used for navigation.